On Weak Observability for Evolution Systems with Skew-Adjoint Generators

Kaïs Ammari, Faouzi Triki

ON WEAK OBSERVABILITY FOR EVOLUTION SYSTEMS WITH SKEW-ADJOINT GENERATORS

KAÏS AMMARI† AND FAOUZI TRIKI‡

Abstract. In the paper we consider the linear inverse problem that consists in recovering the initial state in a first order evolution equation generated by a skew-adjoint operator. We studied the well-posedness of the inversion in terms of the observation operator and the spectra of the skew-adjoint generator. The stability estimate of the inversion can also be seen as a weak observability inequality. The proof of the main results is based on a new resolvent inequality and Fourier transform techniques which are of interest themselves.

Contents

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1. Introduction

Let X be a complex Hilbert space with norm and inner product denoted respectively by $\|\cdot\|_X$ and $\langle\cdot,\cdot\rangle_X$. Let $A : X \to X$ be a linear unbounded self-adjoint, strictly positive operator with a compact resolvent. Denote by $D(A^{\frac{1}{2}})$ the domain of $A^{\frac{1}{2}}$, and introduce for $\beta \in \mathbb{R}$ the scale of Hilbert spaces X_β, as follows: for every $\beta \geqslant 0$, $X_\beta = D(A^{\frac{\beta}{2}})$, with the norm $\|z\|_\beta = \|A^{\frac{\beta}{2}} z\|_X$ (note that $0 \notin \sigma(A)$ where $\sigma(A)$ is the spectrum of A). The space $X_{-\beta}$ is defined by duality with respect to the pivot space X as follows: $X_{-\beta} = X_\beta^*$ for $\beta > 0$.
The operator A can be extended (or restricted) to each X_β, such that it becomes a bounded operator

$$A : X_\beta \to X_{\beta-2} \quad \forall \beta \in \mathbb{R}. \tag{1}$$

The operator iA generates a strongly continuous group of isometries in X denoted $(e^{itA})_{t\in\mathbb{R}}$ [28].

Further, let Y be a complex Hilbert space (which will be identified to its dual space) with norm and inner product respectively denoted by $||.||_Y$ and $\langle\cdot,\cdot\rangle_Y$, and let $C \in \mathcal{L}(X_2, Y)$, the space of linear bounded operators from X_2 into Y.

This paper is concerned with the following abstract infinite-dimensional dual observation system with an output $y \in Y$ described by the equations

$$\left\{ \begin{array}{l} \dot{z}(t) - iAz(t) = 0,\, t > 0, \\ z(0) = z_0 \in X, \\ y(t) = Cz(t),\, t > 0. \end{array} \right. \tag{2}$$

In inverse problems framework the system above is called the direct problem, i.e, to determine the observation $y(t) = Cz(t)$ of the state $z(t)$ for given initial state z_0 and unbounded operator A. The inverse problem is to recover the initial state z_0 from the knowledge of the observation $y(t)$ for $t \in [0, T]$ where $T > 0$ is chosen to be large enough.

Inverse problems for evolution equations driven by numerous applications, have been a very active area in mathematical and numerical research over the last decades [15]. They are intrinsically difficult to solve: this fact is due in part to their very mathematical structure and to the effect that generally only partial data is available. Many different linear inverse problems in evolution equations related to data assimilation, medical imaging, and geoscience, may fit in the general formulation of the system (2) (see for instance [30, 2, 3, 4, 5, 7, 26] and references therein).

The system (2) has a unique weak solution $z \in C(\mathbb{R}, X)$ defined by:

$$z(t) = e^{itA} z_0. \tag{3}$$

If z_0 is not in X_2, in general $z(t)$ does not belong to X_2, and hence the last equation in (2) might not be defined. We further make the following additional admissibility assumption on the observation operator C: $\forall T > 0, \quad \exists C_T > 0,$

$$\forall z_0 \in X_2, \quad \int_0^T \|Ce^{itA} z_0\|_Y^2 dt \leqslant C_T \|z_0\|_X^2. \tag{4}$$

We immediately deduce from the admissibility assumption that the map from X_2 to $L^2_{loc}(\mathbb{R}_+; Y)$ that assigns y for each z_0, has a continuous extension to X. Therefore the last equation in (2) is now well defined for all $z_0 \in X$. Without loss of generality we assume that C_T is an increasing function of T (if the assumption is not satisfied we substitute C_T by $\sup_{0\leqslant t\leqslant T} C_T$).

Since A is a self-adjoint operator with a compact resolvent, it follows that the spectrum of A is given by $\sigma(A) = \{\lambda_k,\ k \in \mathbb{N}^*\}$ where λ_k is a sequence of strictly increasing real numbers. We denote $(\phi_k)_{k\in\mathbb{N}^*}$ the orthonormal sequence of eigenvectors of A associated to the eigenvalues $(\lambda_k)_{k\in\mathbb{N}^*}$.

Let $z \in X_2 \backslash \{0\} \subset X \longmapsto \lambda(z) \in \mathbb{R}_+$ be the A-frequency function defined by

$$\lambda(z) = \langle Az, z\rangle_X \|z\|_X^{-2}, \tag{5}$$

$$= \sum_{k=1}^{+\infty} \lambda_k \langle z, \phi_k\rangle_X^2 \left(\sum_{k=1}^{+\infty} \langle z, \phi_k\rangle_X^2\right)^{-1}. \tag{6}$$

We observe that $z \longmapsto \lambda(z)$ is continuous on $X_2\backslash\{0\}$, and $\lambda(\phi_k) = \lambda_k$, $k \in \mathbb{N}^*$.

Let $\mathfrak{C}$ be the set of functions $\psi : \mathbb{R}_+ \to \mathbb{R}_+^*$ continuous and decreasing. Recall that if $\psi \in \mathfrak{C}$ is not bounded below by a strictly positive constant it satisfies $\lim_{t\to+\infty} \psi(t) = 0$.

Definition 1.1. *The system (2) is said to be weakly observable in time $T > 0$ if there exists $\psi \in \mathfrak{C}$ such that following observation inequality holds:*

$$\forall z_0 \in X_2, \quad \psi(\lambda(z_0))\|z_0\|_X^2 \leqslant \int_0^T \|Ce^{itA} z_0\|_Y^2 dt. \tag{7}$$

If $\psi(t)$ is lower bounded, the system is said to be exactly observable.

Remark 1.1. *If the system (2) is weakly observable in time $T > 0$, it is weakly observable in any time T' larger than T. The function ψ appearing in the observability inequality* (7) *may depends on the time T.*

Most of the existing works on observability inequalities for systems of partial differential equations are based on a time domain techniques as nonharmonic series [1, 16], multipliers method [20, 21], and microlocal analysis techniques [10, 17]. Only few of them have considered frequency domain techniques in the spirit of the well known Fattorini-Hautus test for finite dimensional systems [12, 13, 11, 25, 31].

The wanted frequency domain test for the observability of the system (2) would be only formulated in terms of the operators A, C. The time domain system (2) would be converted into a frequency domain one, and the test would involve essentially the solution in the frequency domain and the observability operator C. The frequency domain test seems to be more suitable for numerical validation and for the calibration of physical models for many reasons: the parameters of the system are in general measured in frequency domain; the computation of the solution is more robust and efficient in frequency domain.

The objective here is to derive sufficient and if possible necessary conditions on

(i) the spectrum of A, and
(ii) on the action of the operator C on the associated eigenfunctions of A,

such that the closed system (2) verifies, for some $T > 0$, sufficiently large, the inequality (7). The aim of this paper is to obtain Fattorini-Hautus type tests on the pair (A, C) that guarantee the *weak observability property* (7).

The rest of the paper is organized as follows: In section 2 we present the main results of our paper related to the weak observability. Section 3 contains the proof of the main Theorem 2.1 based on new resolvent inequality and Fourier transform techniques. In section 4 we study the relation between the spectral coercivity of the observability operator and his action on vector spaces spanned by eigenfunctions associated to close eigenvalues. Finally, in section 5 we apply the results of the main Theorem 2.1 to boundary observability of the Schrödinger equation in a square.

2. Main results

We present in this section the main results of our paper.

Definition 2.1. *The operator C is spectrally coercive if there exist functions $\varepsilon, \psi \in \mathfrak{C}$ such that if $z \in X_2\backslash \{0\}$ satisfies*

$$\frac{\|Az\|_X^2}{\|z\|_X^2} - \lambda^2(z) < \varepsilon(\lambda(z)), \tag{8}$$

then

$$\|Cz\|_Y^2 \geqslant \psi(\lambda(z))\|z\|_X^2. \tag{9}$$

Remark 2.1. *We remark that the following relation*

$$0 \leqslant \|(A - \lambda(z)I)z\|_X^2 \|z\|_X^{-2} = \frac{\|Az\|_X^2}{\|z\|_X^2} - \lambda^2(z) \tag{10}$$

holds for all $z \in X_2 \backslash \{0\}$*. In addition, the equality* $\frac{\|Az\|_X^2}{\|z\|_X^2} - \lambda^2(z) = 0$ *is satisfied if and only if* $z = \phi_k$ *for some* $k \in \mathbb{N}^*$*.*

Now, we are ready to announce our main result.

Theorem 2.1. *The system* (2) *is weakly observable iff* C *is spectrally coercive, that is the following two assertions are equivalent.*

(1) *There exist* $\varepsilon, \psi \in \mathfrak{C}$ *such that if* $z \in X_2 \backslash \{0\}$ *satisfying*

$$0 \leqslant \frac{\|Az\|_X^2}{\|z\|_X^2} - \lambda^2(z) < \varepsilon(\lambda(z)),$$

then

$$\|Cz\|_Y^2 \geqslant \psi(\lambda(z))\|z\|_X^2.$$

(2) *The following weak observation inequality holds:*

$$\forall z_0 \in X_2, \quad \theta_2 \psi\left(\theta_0\left(\frac{1}{T} + \lambda(z_0)\right)\right)\|z_0\|_X^2 \leqslant \int_0^T \|Cz(t)\|_Y^2 d\tau \tag{11}$$

for all $T \geqslant T(\lambda(z_0))$*, where* $T(\lambda(z_0))$ *is the unique solution to the equation*

$$T\varepsilon\left(\theta_0\left(\frac{1}{T} + \lambda(z_0)\right)\right) = \theta_1, \tag{12}$$

and $\varepsilon, \psi \in \mathfrak{C}$ *are the functions appearing in the spectral coercivity of* C*. The strictly positives constants* θ_i*,* $i = 0, 1, 2$*, do not depend on the parameters of the observability system. In addition, the function* $\lambda \mapsto T(\lambda)$ *is increasing.*

The above theorem can be viewed as a extension of several results in the literature [13, 11, 25, 31, 24].

3. Proof of the main Theorem 2.1

In order to prove our main theorem, we need to derive a sequence of preliminary results. We start with the main tool in the proof of the theorem which is a generalized Hautus-type test.

Theorem 3.1. *The operator* $C \in \mathcal{L}(X_2, Y)$*, is spectrally coercive, if and only if there exist functions* ψ*,* $\varepsilon \in \mathfrak{C}$*, such that the following resolvent inequality holds*

$$\|z\|_X^2 \leqslant \inf\left\{\frac{\|Cz\|_Y^2}{\psi(\lambda(z))}, \frac{\|(A - \lambda I)z\|_X^2}{(\lambda - \lambda(z))^2 + \varepsilon(\lambda(z))}\right\}, \ \forall \lambda \in \mathbb{R}, \ \forall z \in X_2 \backslash \{0\}. \tag{13}$$

Proof. Let $z \in X_2 \backslash \{0\}$ be fixed. A forward computation gives the following key identity:

$$\|(A - \lambda I)z\|_X^2 = (\lambda - \lambda(z))^2 \|z\|_X^2 + \|(A - \lambda(z)I)z\|_X^2. \tag{14}$$

We remark that the minimum of $\|(A - \lambda I)z\|_X^2$ for a fixed z with respect to $\lambda \in \mathbb{R}$ is reached at $\lambda = \lambda(z)$.

We first assume that $C \in \mathcal{L}(X_2, Y)$, is spectrally coercive and prove that (13) is satisfied. Let now ε, $\psi \in \mathfrak{C}$ the functions appearing in the spectral coercivity of the operator C in Definition 2.1, and consider the following two possible cases:

(i) The inequality $\|Az\|_X^2 - \lambda^2(z)\|z\|_X^2 < \varepsilon(\lambda(z))\|z\|_X^2$ is satisfied. Then by the spectral coercivity of C, we deduce

$$\|Cz\|_Y^2 \geqslant \psi(\lambda(z))\|z\|_X^2. \tag{15}$$

(ii) The inequality $\|Az\|_X^2 - \lambda^2(z)\|z\|_X^2 \geqslant \varepsilon(\lambda(z))\|z\|_X^2$ holds. Then, the identity (14) implies

$$\|(A-\lambda I)z\|_X^2 \geqslant \left((\lambda-\lambda(z))^2 + \varepsilon(\lambda(z))\right)\|z\|_X^2. \tag{16}$$

By combining both inequalities (15) and (16), we obtain the resolvent inequality (13).

We now assume that (13) holds and, we shall show that $C \in \mathcal{L}(X_2, Y)$, satisfies the spectrally coercivity in Definition 2.1. Let ε, $\psi \in \mathfrak{C}$ the functions appearing in (13), and assume that $z \in X_2\backslash\{0\}$ satisfies

$$\|(A-\lambda(z)I)z\|_X^2 = \|Az\|_X^2 - \lambda^2(z)\|z\|_X^2 < \varepsilon(\lambda(z))\|z\|_X^2. \tag{17}$$

Then, we have two possibilities

(i) The inequality

$$\frac{\|Cz\|_Y^2}{\psi(\lambda(z))} \leqslant \frac{\|(A-\lambda I)z\|_X^2}{(\lambda-\lambda(z))^2 + \varepsilon(\lambda(z))}$$

holds for some $\lambda \in \mathbb{R}$. Consequently the following spectral coercivity

$$\|Cz\|_Y^2 \geqslant \psi(\lambda(z))\|z\|_X^2$$

can be trivially deduced from the resolvent identity (13).

(ii) The inequality

$$\frac{\|Cz\|_Y^2}{\psi(\lambda(z))} > \frac{\|(A-\lambda I)z\|_X^2}{(\lambda-\lambda(z))^2 + \varepsilon(\lambda(z))}$$

is valid for all $\lambda \in \mathbb{R}$. We then deduce from the identity (14) the following inequality

$$\frac{\|Cz\|_Y^2}{\psi(\lambda(z))} > \frac{(\lambda-\lambda(z))^2\|z\|_X^2 + \|(A-\lambda(z)I)z\|_X^2}{(\lambda-\lambda(z))^2 + \varepsilon(\lambda(z))}, \quad \forall \lambda \in \mathbb{R}.$$

Taking λ to infinity we get the wanted inequality, that is

$$\frac{\|Cz\|_Y^2}{\psi(\lambda(z))} \geqslant \|z\|_X^2,$$

which finishes the proof of the Theorem.

□

Next we use a method developed in [11] to derive observability inequalities based on resolvent inequalities and Fourier transform techniques. Our objective is to prove the equivalence between the resolvent inequality (13) and the weak observability (11). The proof of the Theorem is then achieved by considering the results obtained in Theorem 3.1.

We further assume that the resolvent inequality (13) holds and shall prove the weak observability.

Let $\chi \in C_0^\infty(\mathbb{R})$ be a cut off function with a compact support in $(-1,1)$. For $T > 0$, we further denote

$$\chi_T(t) \quad = \quad \chi\left(\frac{t}{T}\right), \qquad t \in \mathbb{R}. \tag{18}$$

Let $z_0 \in X_2\backslash\{0\}$. Set $z(t) = e^{itA}z_0$, $x = \chi_T z$ and $f = \dot{x} - iAx$. Since $\dot{z} - iAz = 0$, we have $f = \dot{\chi}_T z$. The Fourier transform of f with respect to time is given by

$$\widehat{f}(\tau) = (i\tau - iA)\widehat{x}(\tau),$$

where $\widehat{x}(\tau)$ is the Fourier transform of $x(t)$. Applying (13) to $\widehat{x}(\tau) \in X_2\backslash\{0\}$ for $\lambda = \tau$, we obtain

$$\|\widehat{x}(\tau)\|_X^2 \leqslant \inf\left\{\frac{\|C\widehat{x}(\tau)\|_Y^2}{\psi(\lambda(\widehat{x}(\tau)))}, \frac{\|\widehat{f}(\tau)\|_X^2}{(\tau-\lambda(\widehat{x}(\tau)))^2+\varepsilon(\lambda(\widehat{x}(\tau)))}\right\}. \tag{19}$$

We remark that since $\widehat{x}(\tau) \neq 0$, we have $\lambda(\widehat{x}(\tau)) \neq +\infty$, and the inequality (19) is well justified. Next, we study how do the frequency $\lambda(\widehat{x}(\tau))$ behave as a function of τ. We expect that $\lambda(\widehat{x}(\tau))$ that is close to $\lambda(z_0)$, the frequency of the initial state z_0, and reach increases when $|\tau|$ tends to infinity.

To simplify the analysis we will make some assumptions on the cut-off function $\chi(s)$. We further assume that $\chi \in C_0(\mathbb{R})$ satisfies the following inequalities:

$$\chi \in H_0^1(-1,1),\ \frac{\kappa_1}{1+\tau^2} \leqslant |\widehat{\chi}(\tau)| \leqslant \frac{\kappa_2}{1+\tau^2}, \tau \in \mathbb{R}, \tag{20}$$

where $\kappa_2 > \kappa_1 > 0$ are two fixed constants that do not depend on τ. We will show in the Appendix the existence of a such function.

Theorem 3.2. *Let $z_0 \in X_2\backslash\{0\}$, and let $z(t) = e^{itA}z_0$, and let $\widehat{x}(\tau)$ be the Fourier transform of $x(t) = \chi_T(t)z(t)$, where $\chi_T(t)$ is the cut-off function defined by* (18)*, and satisfying the inequality* (20)*.*

Then, there exists a constant $c_0 = c_0(\chi) > 0$ such that the following inequality

$$\lambda_1 \leqslant \lambda(\widehat{x}(\tau)) \leqslant 4|\tau| + c_0\lambda(z_0), \tag{21}$$

holds for all $\tau \in \mathbb{R}$.

Proof. Recall the expression of the frequency function:

$$\lambda(\widehat{x}(\tau)) = \langle A\widehat{x}(\tau), \widehat{x}(\tau)\rangle_X \|\widehat{x}(\tau)\|_X^{-2},\ \forall \tau \in \mathbb{R}. \tag{22}$$

Let $z_0 = \sum_{k=1}^{+\infty} z_k\phi_k \in X_2$. Hence

$$\widehat{x}(\tau) = \sum_{k=1}^{+\infty} \widehat{\chi}_T(\tau-\lambda_k)z_k\phi_k. \tag{23}$$

Hence

$$\lambda(\widehat{x}(\tau)) = \sum_{k=1}^{+\infty} \lambda_k|\widehat{\chi}_T(\tau-\lambda_k)|^2 z_k^2 \left(\sum_{k=1}^{+\infty} |\widehat{\chi}_T(\tau-\lambda_k)|^2 z_k^2\right)^{-1}. \tag{24}$$

We first remark that $\lambda(\widehat{x}(\tau)) \geqslant \lambda_1$ for all $\tau \in \mathbb{R}$, and it tends to $\lambda(z_0)$ when T approaches 0. In order to study the behavior of $\lambda(\widehat{x}(\tau))$ when τ is large we need to derive the behavior of $\widehat{\chi}_T(s)$ when s tends to infinity.

We start with the trivial case where τ is far away from the spectrum of A, that is $\tau < \lambda_1$.

Let $K \in \mathbb{R}_+$ be large enough, and set

$$\sum_{k=1}^{+\infty} |\widehat{\chi}_T(\tau-\lambda_k)|^2 z_k^2 = \sum_{|\tau-\lambda_k|\leqslant K} |\widehat{\chi}_T(\tau-\lambda_k)|^2 z_k^2 + \sum_{|\tau-\lambda_k|>K} |\widehat{\chi}_T(\tau-\lambda_k)|^2 z_k^2 = \mathcal{I}_1 + \mathcal{I}_2.$$

We claim that there exists $K_\tau > 0$ large enough such that

$$2\mathcal{I}_2 \leqslant \mathcal{I}_1, \quad \text{for all } K \geqslant K_\tau. \tag{25}$$

We first observe that there exists $r_0 > 0$ large enough such that

$$2\kappa_2 \sum_{\lambda_k > r_0} z_k^2 \leqslant \kappa_1 \sum_{\lambda_k \leqslant r_0} z_k^2, \tag{26}$$

or equivalently

$$\left(2\frac{\kappa_2}{\kappa_1}+1\right)\sum_{\lambda_k>r_0} z_k^2 \leqslant \|z_0\|_X^2.$$

In fact, we have

$$\sum_{\lambda_k>r_0} z_k^2 < \frac{1}{r_0}\sum_{\lambda_k>r_0}\lambda_k z_k^2 \leqslant \frac{\lambda(z_0)}{r_0}\|z_0\|_X^2. \tag{27}$$

Hence the inequality (26) holds if

$$r_0 = 2\left(2\frac{\kappa_2}{\kappa_1}+1\right)\lambda(z_0). \tag{28}$$

Now by taking $K=|\tau|+r_0$, and using the bounds (20) with $\widehat{\chi}_T(s)=T\widehat{\chi}(Ts)$ in mind, we get

$$2\mathcal{I}_2 \leqslant \frac{2\kappa_2 T^2}{(1+K^2T^2)^2}\sum_{\lambda_k>K+\tau} z_k^2, \tag{29}$$

$$\mathcal{I}_1 \geqslant \frac{\kappa_1 T^2}{(1+K^2T^2)^2}\sum_{\lambda_k\leqslant K+\tau} z_k^2. \tag{30}$$

Since $K\geqslant r_0$, inequalities (26), (29) and (30) imply

$$2\mathcal{I}_2 \leqslant \frac{\kappa_1 T^2}{(1+K^2T^2)^2}\sum_{\lambda_k\leqslant K+\tau}\lambda_k z_k^2 \leqslant \mathcal{I}_1. \tag{31}$$

Then, inequality (25) is valid for $K_\tau=\max(\tau,r_0)$. Consequently the inequalities

$$\frac{1}{2}\mathcal{I}_1 \leqslant \|\widehat{x}(\tau)\|_X^2 = \sum_{k=1}^{+\infty}|\widehat{\chi}_T(\tau-\lambda_k)|^2 z_k^2 \leqslant 3\mathcal{I}_1, \tag{32}$$

holds for all $K\geqslant K_\tau$.

Considering now identity (24), and inequalities (32), we obtain

$$\lambda(\widehat{x}(\tau)) \leqslant 2\left(\sum_{|\tau-\lambda_k|\leqslant K}\lambda_k|\widehat{\chi}_T(\tau-\lambda_k)|^2 z_k^2\right)\left(\sum_{|\tau-\lambda_k|\leqslant K}|\widehat{\chi}_T(\tau-\lambda_k)|^2 z_k^2\right)^{-1}$$

$$+2\left(\sum_{|\tau-\lambda_k|>K}\lambda_k|\widehat{\chi}_T(\tau-\lambda_k)|^2 z_k^2\right)\left(\sum_{|\tau-\lambda_k|\leqslant K}|\widehat{\chi}_T(\tau-\lambda_k)|^2 z_k^2\right)^{-1} = \mathcal{J}_1+\mathcal{J}_2.$$

On the other hand we have

$$\mathcal{J}_1 \leqslant 2(\tau+K). \tag{33}$$

In addition, using again the bounds (20), we obtain

$$\mathcal{J}_2 \leqslant 2\left(\sum_{\lambda_k>\tau+K}\lambda_k z_k^2\right)\left(\sum_{\lambda_k\leqslant K+\tau} z_k^2\right)^{-1}. \tag{34}$$

Since $K+\tau\geqslant r_0$, inequality (26) gives

$$\sum_{\lambda_k\leqslant K+\tau} z_k^2 \geqslant \left(\frac{\kappa_1}{2\kappa_2}+1\right)^{-1}\|z_0\|_X^2. \tag{35}$$

Hence

$$\mathcal{J}_2 \leqslant 2\left(\frac{\kappa_1}{2\kappa_2}+1\right)\left(\sum_{k=1}^{+\infty}\lambda_k z_k^2\right)\left(\sum_{k=1}^{+\infty} z_k^2\right)^{-1} = 2\left(\frac{\kappa_1}{2\kappa_2}+1\right)\lambda(z_0). \tag{36}$$

Combining inequalities (33), (36)and (35), we get

$$\lambda(\widehat{x}(\tau)) \leqslant 2|\tau| + 2K + 2\left(\frac{\kappa_1}{2\kappa_2}+1\right)\lambda(z_0).$$

for all $K \geqslant K_\tau$.

Consequently, the proof is achieved by taking $c_0 = 8\frac{\kappa_2}{\kappa_1} + \frac{\kappa_1}{\kappa_2} + 6$.

□

Remark 3.1. *The upper bound of $\lambda(\widehat{x}(\tau))$ obtained in Theorem 3.2 is not optimal since $\lambda(\widehat{x}(\tau)) = \lambda_k = \lambda(z_0)$ if $z_0 = \phi_k$. Moreover when $\lambda_{max}(z_0) = \max\{\lambda_k,\ k \in \mathbb{N}^*,\ \langle z_0, \phi_k\rangle_X \neq 0\} < \infty$, we can easily show that $\lambda(\widehat{x}(\tau)) \leqslant \lambda_{max}(z_0)$. We remark that in both cases the bounds of $\lambda(\widehat{x}(\tau))$ are independent of the Fourier frequency τ.*

Lemma 3.1. *Let $c_0' = \frac{\|\dot{\chi}\|_{L^2(-1,1)}}{\|\chi\|_{L^2(-1,1)}}$, $z_0 \in X_2\backslash\{0\}$, and let $z(t) = e^{itA}z_0$, and let $\widehat{x}(\tau)$ be the Fourier transform of $x(t) = \chi_T(t)z(t)$, where $\chi_T(t)$ is the cut-off function defined by (18).*

Then, the following inequality

$$\left(1 - \frac{1}{R}\left(\frac{c_0'}{T} + \lambda(z_0)\right)\right)\|z_0\|_X^2 \leqslant \|\chi\|_{L^2(-1,1)}^{-2}\int_{-R}^{R}\|\widehat{x}(\tau)\|_X^2 d\tau \tag{37}$$

holds for all $R > \frac{c_0'}{T} + \lambda(z_0)$.

Proof. Recall that $\dot{x} = f + iAx$ where $f = \dot{\chi}_T z$. By integration by parts we then have

$$\widehat{x}(\tau) = -\frac{i}{\tau}\left(\widehat{f}(\tau) + iA\widehat{x}(\tau)\right).$$

Consequently

$$\|\widehat{x}(\tau)\|_X^2 = \langle -\frac{i}{\tau}\left(\widehat{f}(\tau) + iA\widehat{x}(\tau)\right), \widehat{x}(\tau)\rangle_X.$$

Then for any $R > 0$, by Fourier-Plancherel Theorem, we have

$$\|\chi\|_{L^2(-1,1)}^2\|z_0\|_X^2 \leqslant \int_{-R}^{R}\|\widehat{x}(\tau)\|_X^2 d\tau + \frac{1}{R}\left(\frac{1}{T}\|\dot{\chi}\|_{L^2(-1,1)}\|\chi\|_{L^2(-1,1)} + \lambda(z_0)\|\chi\|_{L^2(-1,1)}^2\right)\|z_0\|_X^2.$$

Hence for R large enough we have

$$\left(1 - \frac{1}{R}\left(\frac{1}{T}\frac{\|\dot{\chi}\|_{L^2(-1,1)}}{\|\chi\|_{L^2(-1,1)}} + \lambda(z_0)\right)\right)\|z_0\|_X^2 \leqslant \|\chi\|_{L^2(-1,1)}^{-2}\int_{-R}^{R}\|\widehat{x}(\tau)\|_X^2 d\tau,$$

which finishes the proof of the lemma.

□

Back now to the proof of the theorem. Combining inequalities (19) and (37), we find

$$\left(1 - \frac{1}{R}\left(\frac{c_0}{T} + \lambda(z_0)\right)\right)\|z_0\|_X^2 \leqslant \|\chi\|_{L^2(-1,1)}^{-2}\left(\int_{-R}^{R}\frac{\|C\widehat{x}(\tau)\|_Y^2}{\psi(\lambda(\widehat{x}(\tau)))}d\tau + \int_{-R}^{R}\frac{\|\widehat{f}(\tau)\|_X^2}{\varepsilon(\lambda(\widehat{x}(\tau)))}d\tau\right). \tag{38}$$

Applying the upper bound $\lambda(\widehat{x}(\tau))$ derived in Theorem 3.2, and considering the monotony of the functions ψ and ε in $\mathfrak{C}$, we obtain

$$\left(1-\frac{1}{R}\left(\frac{c_0'}{T}+\lambda(z_0)\right)\right)\|z_0\|_X^2 \leqslant \frac{1}{\psi(4R+c_0\lambda(z_0))}\frac{\|\chi\|^2_{L^\infty(-1,1)}}{\|\chi\|^2_{L^2(-1,1)}}\int_0^T |Cz(t)\|_Y^2 d\tau$$
$$+\frac{1}{T\varepsilon(4R+c_0\lambda(z_0))}\frac{\|\dot{\chi}\|^2_{L^2(-1,1)}}{\|\chi\|^2_{L^2(-1,1)}}\|z_0\|_X^2,$$

for all $R>\frac{c_0'}{T}+\lambda(z_0)$.

Now, by taking $R=2\left(\frac{c_0}{T}+\lambda(z_0)\right)$, and $\theta_0=\max(c_0',8+c_0)$, we find

$$\left(1-\frac{2}{T\varepsilon\left(\theta_0\left(\frac{1}{T}+\lambda(z_0)\right)\right)}\frac{\|\dot{\chi}\|^2_{L^2(-1,1)}}{\|\chi\|^2_{L^2(-1,1)}}\right)\|z_0\|_X^2 \leqslant \frac{2}{\psi\left(\theta_0\left(\frac{1}{T}+\lambda(z_0)\right)\right)}\frac{\|\chi\|^2_{L^\infty(-1,1)}}{\|\chi\|^2_{L^2(-1,1)}}\int_0^T \|Cz(t)\|_Y^2 dt.$$

Let $\theta_1=\frac{4\|\chi\|^2_{L^2(-1,1)}}{\|\dot{\chi}\|^2_{L^\infty(-1,1)}}$, and $\theta_2=\frac{4\|\chi\|^2_{L^2(-1,1)}}{\|\chi\|^2_{L^\infty(-1,1)}}$.

Then, for $T\varepsilon(4R+c_0\lambda(z_0))\geqslant\theta_1$, we finally get the wanted estimate:

$$\theta_2\psi\left(\theta_0\left(\frac{1}{T}+\lambda(z_0)\right)\right)\|z_0\|_X^2\leqslant\int_0^T\|Cz(t)\|_Y^2dt. \tag{39}$$

Simple calculation shows that the function $T\mapsto T\varepsilon\left(\theta_0\left(\frac{1}{T}+\lambda(z_0)\right)\right)$ is increasing, tends to infinity when T approaches $+\infty$, and tends to 0 when T approaches 0. Then there exists a unique value $T(\lambda(z_0))>0$ that solves the equation (12). In addition, the function $\lambda\mapsto T(\lambda)$ is increasing. Finally, the inequality (39) is valid for all $T\geqslant T(\lambda(z_0))$.

Now, we shall prove the converse. Our strategy is to adapt the proof of Theorem 1.2 in [29] for the classical exact controllability to our settings (see also [11, 24]). We further assume that the weak observability inequality (11) holds for some fixed ψ and ε in $\mathfrak{C}$. Our goal now is to show that C is indeed spectrally coercive.

Let $z_0\in X_4$, and $x_0:=(iA-i\tau I)z_0$ for some $\tau\in\mathbb{R}$. Define $x(t)=e^{itA}x_0$ and $z(t)=e^{itA}z_0$.

A forward computation shows that $z(t)$ solves the following

$$\begin{aligned}\dot{z}(t)-i\tau z(t)&=x(t),\quad\forall t\in\mathbb{R}_+^*,\\ z(0)&=z_0.\end{aligned}$$

Then

$$z(t)=e^{i\tau t}z_0+\int_0^t e^{i\tau(t-s)}x(s)ds.$$

Applying now the observability operator both sides gives

$$Cz(t)=e^{i\tau t}Cz_0+\int_0^t e^{i\tau(t-s)}Cx(s)ds,$$

whence

$$\|Cz(t)\|_Y^2\leqslant2\|Cz_0\|_Y^2+2\int_0^t\|Cx(s)\|_Y^2ds.$$

Integrating the inequality above both sides over $(0,T)$, we obtain

$$\int_0^T \|Cz(t)\|_Y^2 dt \leqslant 2T\|Cz_0\|_Y^2 + 2T\int_0^T \|Cx(s)\|_Y^2 ds.$$

We deduce from the admissibility assumption (4) that

$$\int_0^T \|Cz(t)\|_Y^2 dt \leqslant 2T\|Cz_0\|_Y^2 + 2TC_T\|(A-\tau I)z_0\|_X^2.$$

Applying the weak observability inequality (11) for $T = T(\lambda(z_0))$, leads to

$$\theta_2\psi\left(\theta_0\left(\frac{1}{T(\lambda(z_0))} + \lambda(z_0)\right)\right)\|z_0\|_X^2 \leqslant 2T(\lambda(z_0))\|Cz_0\|_Y^2 + 2T(\lambda(z_0))C_{T(\lambda(z_0))}\|(A-\tau I)z_0\|_X^2,$$

for all $\tau \in \mathbb{R}$.

Since $T(\lambda) \geqslant T_0 = T(0)$, for all $\lambda \geqslant 0$, we have

$$\theta_2\psi\left(\theta_0\left(\frac{1}{T_0} + \lambda(z_0)\right)\right)\|z_0\|_X^2 \leqslant 2T(\lambda(z_0))\|Cz_0\|_Y^2 + 2T(\lambda(z_0))C_{T(\lambda(z_0))}\|(A-\tau I)z_0\|_X^2,$$

Taking $\tau = \lambda(z_0)$ in the previous inequality implies

$$\frac{\theta_2}{2T(\lambda(z_0))C_{T(\lambda(z_0))}}\psi\left(\theta_0\left(\frac{1}{T_0} + \lambda(z_0)\right)\right)\|z_0\|_X^2 \leqslant \frac{1}{C_{T(\lambda(z_0))}}\|Cz_0\|_X^2 + \|(A-\lambda(z_0)I)z_0\|_X^2.$$

Let

$$\widetilde{\psi}(\lambda) = \frac{\theta_2}{4T(\lambda)}\psi\left(\theta_0\left(\frac{1}{T_0} + \lambda\right)\right),$$
$$\widetilde{\varepsilon}(\lambda) = \frac{\theta_2}{4T(\lambda)C_\lambda}\psi\left(\theta_0\left(\frac{1}{T_0} + \lambda\right)\right).$$

We deduce from the monotonicity properties of $\psi(\lambda), C_\lambda$, and $T(\lambda)$ that $\widetilde{\psi}(\lambda)$, $\widetilde{\varepsilon}(\lambda) \in \mathfrak{C}$.

Consequently C becomes spectrally coercive with the functions $\widetilde{\psi}(\lambda)$, $\widetilde{\varepsilon}(\lambda)$, that is

$$0 \leqslant \frac{\|Az\|_X^2}{\|z\|_X^2} - \lambda^2(z) < \widetilde{\varepsilon}(\lambda(z)),$$

implies

$$\|Cz\|_Y^2 \geqslant \widetilde{\psi}(\lambda(z))\|z\|_X^2,$$

which finishes the proof of the Theorem.

4. Sufficient conditions for the spectral coercivity.

In this section we study the relation between the spectral coercivity of the observability operator C given in Definition 2.1, and the action of the operator C on vector spaces spanned by eigenfunctions associated to close eigenvalues.

For $\lambda \in \mathbb{R}_+$ and $\varepsilon > 0$, set

$$N_\varepsilon(\lambda) = \{k \in \mathbb{N}^* \text{ such that } |\lambda - \lambda_k| < \varepsilon\}, \tag{40}$$

to be the index function of eigenvalues of A in a ε-neighborhood of a given λ.

Definition 4.1. *The operator C is weakly spectrally coercive if there exist a constant $\varepsilon > 0$ and a function $\psi \in \mathfrak{C}$ such that for all $\lambda \in \mathbb{R}$, the following inequality*

$$\|Cz\|_Y^2 \geqslant \psi(\lambda)\|z\|_X^2, \tag{41}$$

holds for all $z = \sum_{k\in N_\varepsilon(\lambda)} z_k\phi_k \in X_2\backslash\{0\}$.

Lemma 4.1. *The operator C is weakly spectrally coercive iff there exist a constant $\varepsilon > 0$ and a function $\psi \in \mathfrak{C}$ such that the following inequality*

$$\|Cz\|_Y^2 \geqslant \psi(\lambda_n)\|z\|_X^2, \tag{42}$$

holds for all $z = \sum_{k\in N_\varepsilon(\lambda_n)} z_k\phi_k$, and for all $n \in \mathbb{N}^*$.

Proof. Assume that C is weakly spectrally coercive. By taking $\lambda = \lambda_n$ in (41), inequality (42) immediately holds. Conversely, assume that inequality (42) is satisfied, and let $\lambda \in \mathbb{R}$. One can easily check that the set $N_{\frac{\varepsilon}{2}}(\lambda)$ is either empty or it contains at least an element $n_0 \in \mathbb{N}^*$. Since $N_{\frac{\varepsilon}{2}}(\lambda) \subset N_\varepsilon(\lambda_{n_0})$, we have

$$\|Cz\|_Y^2 \geqslant \psi(\lambda_{n_0})\|z\|_X^2,$$

holds for all $z = \sum_{k\in N_{\frac{\varepsilon}{2}}(\lambda)} z_k\phi_k \in X_2\backslash\{0\}$. On the other hand the fact that ψ is non-increasing implies

$$\|Cz\|_Y^2 \geqslant \psi\left(\lambda + \frac{\varepsilon}{2}\right)\|z\|_X^2,$$

holds for all $z = \sum_{k\in N_{\frac{\varepsilon}{2}}(\lambda)} z_k\phi_k \in X_2\backslash\{0\}$, which shows that C is weakly spectrally coercive with the constant $\frac{\varepsilon}{2} > 0$ and $\tilde{\psi}(\lambda) := \psi(\lambda + \frac{\varepsilon}{2}) \in \mathfrak{C}$. □

The Lemma 4.1 has been proved in [25] for the particular case where ψ is a constant function.

Theorem 4.1. *Let $\varepsilon > 0$ be a fixed constant and let $\psi \in \mathfrak{C}$. If C is spectrally coercive with ε, ψ, then it is weakly spectrally coercive. Conversely, if C is weakly spectrally coercive with ε, ψ, then C is spectrally coercive.*

Proof. Let $\lambda \in \mathbb{R}_+$, and $\beta > 0$ being fixed. A direct calculation shows that if

$$z = \sum_{k\in N_\beta(\lambda)} z_k\phi_k,$$

we have

$$(\lambda(z) - \lambda)\|z\|_X^2 = \sum_{k\in N_\beta(\lambda)} (\lambda_k - \lambda)z_k^2.$$

Hence

$$|\lambda(z) - \lambda| < \beta.$$

On the other hand

$$\|Az\|_X^2 - \lambda^2(z)\|z\|_X^2 = \|(A - \lambda(z)I)z\|_X^2 \leqslant 2\|(A - \lambda I)z\|_X^2 + 2|\lambda - \lambda(z)|^2\|z\|_X^2 < 2\beta^2\|z\|_X^2.$$

Then, we deduce from the spectral coercivity in Definition 2.1 that (41) holds if we choose β such that $2\beta^2 < \varepsilon$.

Now, we shall prove the opposite implication. Assume that (41) is satisfied for all $\lambda \in \mathbb{R}_+$, and let

$$z = \sum_{k=1}^{+\infty} z_k\phi_k,$$

being in $X_2\backslash\{0\}$, and satisfying the inequality

$$\frac{\|Az\|_X^2}{\|z\|_X^2} - \lambda^2(z) < \beta(\lambda(z)), \tag{43}$$

where β will be chosen later in terms of ε and ψ.

Set

$$((A - \lambda(z)I)z = f. \tag{44}$$

We deduce from (43), the following estimate

$$\|f\|_X^2 \leqslant \beta\|z\|_X^2. \tag{45}$$

We now introduce the following orthogonal decomposition of z:

$$z \quad = \quad z^0 + \widetilde{z}, \tag{46}$$

with

$$z^0 = \sum_{k \in N_\varepsilon(\lambda(z))} z_k \phi_k, \quad \widetilde{z} = \sum_{k \notin N_\varepsilon(\lambda(z))} z_k \phi_k. \tag{47}$$

We deduce from (43), (44) and (45) the following estimate

$$\|\widetilde{z}\|_X^2 = \sum_{k \notin N_\varepsilon(\lambda(z))} z_k^2 = \sum_{k \notin N_\varepsilon(\lambda(z))} \frac{f_k^2}{(\lambda(z) - \lambda_k)^2} \leqslant \frac{1}{\varepsilon^2}\|f\|_X^2 \leqslant \frac{\beta}{\varepsilon^2}\|z\|_X^2. \tag{48}$$

On the other hand the inequality (41) for $\lambda = \lambda(z)$ implies

$$\|z^0\|_X^2 \leqslant \frac{\|Cz^0\|_Y^2}{\psi(\lambda(z))}. \tag{49}$$

The following result has been proved for admissible operator C first on $(0, +\infty)$ in [29], and on $(0, T)$ in [25].

Proposition 4.1. *For each $\varepsilon > 0$ and $\lambda \in \mathbb{R}_+$, we define the subspace $V(\lambda) \subset X$ by*

$$V(\lambda) := \{\phi_k : \ k \notin N_\varepsilon(\lambda)\},$$

and we denote $A_\lambda : V(\lambda) \cap X_2 \to X$, the restriction of the unbounded operator to $V(\lambda)$.

Then, there exists a constant $M > 0$, such that

$$\|C(A_\lambda - \lambda I)^{-1}\|_{\mathcal{L}(V(\lambda), Y)} \leqslant M, \quad \forall \lambda \in \mathbb{R}_+. \tag{50}$$

We deduce from (44) and (46), the following inequality

$$\|Cz^0\|_Y^2 \leqslant 2\|Cz\|_Y^2 + 2\|C\widetilde{z}\|_Y^2 \leqslant 2\|Cz\|_Y^2 + 2\|C(A_{\lambda(z)} - \lambda(z)I)^{-1}f\|_Y^2. \tag{51}$$

Applying now the results of Proposition 4.1 on (51), we get

$$\|Cz^0\|_Y^2 \leqslant 2\|Cz\|_Y^2 + 2M\|f\|_X^2. \tag{52}$$

Inequalities (45) and (52), give

$$\|Cz^0\|_Y^2 \leqslant 2\|Cz\|_Y^2 + 2M\beta\|z\|_X^2.$$

Now, using the inequality (49), we get

$$\|z^0\|_X^2 \leqslant 2\frac{\|Cz\|_Y^2}{\psi(\lambda(z))} + \frac{2M\beta}{\psi(\lambda(z))}\|z\|_X^2. \tag{53}$$

Combining (48) and (53), we obtain

$$\|z\|_X^2 = \|z^0\|_X^2 + \|\tilde{z}\|_X^2 \leqslant \rho(\lambda(z))\|z\|_X^2 + 2\frac{\|Cz\|_Y^2}{\psi(\lambda(z))},$$

with

$$\rho(\lambda(z)) := \left(\frac{2M}{\psi(\lambda(z))} + \frac{1}{\varepsilon}\right)\beta(\lambda(z)). \tag{54}$$

By taking

$$\beta(\lambda(z)) := \frac{1}{2}\left(\frac{2M}{\psi(\lambda(z))} + \frac{1}{\varepsilon}\right)^{-1}, \tag{55}$$

we find

$$\frac{1}{4}\,\psi(\lambda(z))\|z\|_X^2 \leqslant \|Cz\|_Y^2.$$

One can check easily that $\beta(\lambda)$ belongs to $\mathfrak{C}$. Then C becomes spectrally coercive with the functions $\beta(\lambda), \frac{1}{4}\,\psi(\lambda) \in \mathfrak{C}$.

□

Remark 4.1. *Theorem 4.1 shows that the results of the paper* [25] *by M. Tucsnak and al. correspond to the particular case of spectral coercivity where* ε *and* ψ *are constant functions. Finally, applying Proposition 4.1 is not necessary to prove the theorem. In fact we can bound in inequality* (51)*,* C *by* $\|C\|^2(\lambda(z)+\varepsilon)^2\frac{\beta}{\varepsilon^2}\|z\|_X^2$ *where* $\|C\|$ *is the norm of* C *in* $\mathcal{L}(X_2, Y)$*. Applying the results of Proposition 4.1 improves the behavior of* $\varepsilon(\lambda)$ *for large* λ*.*

5. Application to observability of the Schrödinger equation

Let $\Omega = (0,\pi)\times(0,\pi)$, and $\partial\Omega$ be its boundary. We consider the following initial and boundary value problem:

$$\begin{cases} z'(x,t) + i\Delta z(x,t) = 0,\ x \in \Omega,\ t > 0, \\ z(x,t) = 0,\ x \in \partial\Omega,\ t > 0, \\ z(x,0) = z_0(x), x \in \Omega. \end{cases} \tag{56}$$

Let Γ be an open nonempty subset of $\partial\Omega$. Define C to be the following boundary observability operator

$$y(x,t) = Cz(x,t) = \partial_\nu z|_\Gamma, \tag{57}$$

where ν is the outward normal vector on $\partial\Omega$, and ∂_ν is the normal derivative.

We further show that the observation system (56)-(57) fits perfectly in the general formulation of the system (2).

Let $X = H_0^1(\Omega)$ be the Hilbert space with scalar product

$$\langle v, w\rangle_X = \int_\Omega \nabla u\cdot\nabla v\,dx.$$

Therefore $A = -\Delta : X_2 \subset X \to X$, is a linear unbounded self-adjoint, strictly positive operator with a compact resolvent. Hence the operator iA generates a strongly continuous group of isometries in X denoted $(e^{itA})_{t\in\mathbb{R}}$. Moreover for $\beta \geqslant 0$, $X_\beta = D(A^{\frac{\beta}{2}})$ is given by

$$X_\beta = \left\{\phi \in H_0^1(\Omega) : (-\Delta)^{\frac{\beta}{2}}\phi \in H_0^1(\Omega)\right\}.$$

Then the observability operator $C : X_2 \to Y := L^2(\Gamma)$, defined by (57), is a bounded operator. In addition it is known that C is an admissible observability operator, that is for any $T > 0$ there exists a constant $C_T > 0$, such that the following inequality holds

$$\int_0^T \int_\Gamma |\partial_\nu z|^2 \, ds(x)dt \leqslant C_T^2 \int_\Omega |\nabla z_0|^2 \, dx,$$

for all $z_0 \in X_2$.

The eigenvalues of A are

$$\lambda_{m,n} = m^2 + n^2, \quad m, n \in \mathbb{N}^*. \tag{58}$$

A corresponding family of normalized eigenfunctions in $H_0^1(\Omega)$ are

$$\phi_{m,n}(x) = \frac{2}{\pi\sqrt{m^2+n^2}} \sin(n\pi x_1)\sin(m\pi x_2), \quad m, n \in \mathbb{N}^*, \quad x = (x_1, x_2) \in \Omega. \tag{59}$$

Next we derive observability inequalities corresponding to different geometrical assumptions on the observability set Γ.

Assumption **I**: We assume that Γ contains at least two touching sides of Ω.

In this case it is known that Γ satisfies the geometrical assumptions of [10], and the exact controllability is reached [19]. We will show that it is indeed the situation by applying our coercivity test.

Consider the Helmholtz equation defined by

$$\begin{cases} \Delta u + k^2 u = f, \, x \in \Omega, \\ u = 0, \, x \in \partial\Omega \setminus \overline{\Gamma}, \\ \partial_\nu u - iku = 0, \, x \in \Gamma, \end{cases} \tag{60}$$

where $g \in L^2(\Gamma)$ and $f \in L^2(\Omega)$.

It has been shown using Rellich's identities (which are somehow related to the multiplier approach in observability [20, 21]) the following result [14].

Proposition 5.1. *Under the assumptions* **I** *on* Γ*, a solution* $u \in H^1(\Omega)$ *to the system* (60) *satisfies the following inequality*

$$k\|u\|_{L^2(\Omega)} + \|\nabla u\|_{L^2(\Omega)} \leqslant c_0 \left(\|f\|_{L^2(\Omega)} + \|g\|_{L^2(\Gamma)}\right), \tag{61}$$

for all $k \geqslant k_0$*, where* $k_0 > 0$ *and* $c_0 > 0$ *are constants that only depend on* Γ*.*

We deduce from Proposition 5.1 the following inequality

$$\|z\|_X \leqslant c_1 \left(\|Az - \lambda(z)z\|_X + \|Cz\|_Y\right), \tag{62}$$

for all $z \in X_2 \setminus \{0\}$, where $\lambda(z)$ is the A-frequency of z, and $c_1 > 0$ are constants that only depend on Γ. Then by taking $\varepsilon(\lambda) = \frac{1}{4c_1^2}$, we find that C is spectrally coercive with $\psi(\lambda) = \frac{1}{4c_1^2}$, which implies in turn that the system (56)-(57) is exactly observable.

Theorem 5.1. *Under the assumptions* **I** *on* Γ*, the system* (56)-(57)*, is exactly observable.*

Assumption **II**: We assume that Γ in a one side of Ω. Without loss of generality, we further assume that $\Gamma = (0, \pi) \times \{0\}$.

The following result has been derived partially in [8].

Proposition 5.2. *Under the assumptions* **II** *on* Γ*, a solution* $u \in H^1(\Omega)$ *to the system* (60) *satisfies the following inequality*

$$k\|u\|_{L^2(\Omega)} + \|\nabla u\|_{L^2(\Omega)} \leqslant c_0 k\left(\|f\|_{L^2(\Omega)} + \|g\|_{L^2(\Gamma)}\right), \tag{63}$$

for all $k \geqslant k_0$*, where* $k_0 > 0$ *and* $c_0 > 0$ *are constants that only depend on* Γ*.*

We again deduce from Proposition 5.2 the following resolvent inequality

$$\|z\|_X \leqslant c_1(1 + \sqrt{\lambda(z)})\left(\|Az - \lambda(z)z\|_X + \|Cz\|_Y\right), \tag{64}$$

for all $z \in X_2\backslash\{0\}$, where $\lambda(z)$ is the A-frequency of z, and $c_1 > 0$ is a constant that only depends on Γ. Then by taking $\varepsilon(\lambda) = \frac{1}{8c_1^2(1+\lambda)}$, we find that C is spectrally coercive with $\psi(\lambda) = \frac{1}{8c_1^2(1+\lambda)}$. This implies in turn that the system (56)-(57) is weakly observable: there exists a constant $T^0 > 0$ such that

$$\psi(\lambda(z_0))\|z_0\|^2_{H_0^1(\Omega)} \leqslant \int_0^T \int_\Gamma |\partial_\nu z|^2 \, ds(x)dt, \tag{65}$$

for all $z_0 \in X_2$, and for all $T \geqslant T^0$.

Theorem 5.2. *Under the assumptions* **II** *on* Γ*, the system* (56)-(57)*, is weakly observable for any* $z_0 \in X$*.*

Assumption **III**: We assume that $\overline{\Gamma}$ is included in a one side of Ω. Without loss of generality, we further assume that $(\alpha, \beta) \times \{0\} \subset \Gamma \subset (0, \pi) \times \{0\}$, with $0 < \alpha < \beta < \pi$. Then, we have the following weak observability inequality.

Theorem 5.3. *Under the assumptions* **III** *on* Γ*, the system* (56)-(57)*, is weakly observable for any* $z_0 \in X$ *with* $\widetilde{\varepsilon}(\lambda) = \frac{1}{\frac{4M}{\delta_\Gamma}\lambda + 1}$ *and* $\widetilde{\psi}(\lambda) = \frac{\delta_\Gamma}{4\lambda}$*, where* $\delta_\Gamma > 0$ *is a constant that only depends on* Γ*, and* $M > 0$ *is the admissibility constant appearing in Proposition 4.1.*

Different from the proofs in the two first cases, the proof of the weak observability in the theorem above is based on intrinsic properties of the eigenelements of A and the operator C. We first present the following useful result.

Lemma 5.1. *The operator* C *is weakly spectrally coercive, that is, the following inequality*

$$\|Cz\|^2_Y \geqslant \psi(\lambda_{m,n})\|z\|^2_X, \tag{66}$$

holds for all $z = \sum_{k \in N_{\frac{1}{2}}(\lambda_{m,n})} z_k\phi_k$ *where* $\psi(\lambda) = \frac{\delta_\Gamma}{\lambda}$*, with* $\delta_\Gamma > 0$ *is a constant that only depends on* Γ*.*

Proof. Let $\lambda_{m,n} = m^2 + n^2$ be fixed eigenvalue, and let $z = \sum_{k \in N_{\frac{1}{2}}(\lambda_{m,n})} z_k\phi_k$ be a fixed vector in $X_2\backslash\{0\}$.

It is easy to check that

$$N_{\frac{1}{2}}(\lambda_{m,n}) = \{k = (p, q) \in \mathbb{N}^* \times \mathbb{N}^* : p^2 + q^2 = m^2 + n^2\}. \tag{67}$$

Therefore

$$\begin{aligned} \|Cz\|^2_Y &= \int_\Gamma \left| \sum_{k \in N_{\frac{1}{2}}(\lambda_{m,n})} z_k C\phi_k(x) \right|^2 ds(x), \\ &\geqslant \frac{4}{\pi^2}\int_\alpha^\beta \left| \sum_{p^2+q^2=m^2+n^2} \frac{q}{(p^2+q^2)^{\frac{1}{2}}} z_{p,q} \sin(px_1) \right|^2 dx_1, \end{aligned} \tag{68}$$

Based on techniques related to nonharmonic Fourier series, the following inequality has been proved in Proposition 7 of [25].

$$\int_{\alpha}^{\beta} \left| \sum_{p^2+q^2=m^2+n^2} \frac{q}{(p^2+q^2)^{\frac{1}{2}}} z_{p,q} \sin(px_1) \right|^2 dx_1 \geqslant \tilde{\delta}_{\alpha,\beta} \sum_{p^2+q^2=m^2+n^2} \frac{q^2}{p^2+q^2} |z_{p,q}|^2, \tag{69}$$

where $\tilde{\delta}_{\alpha,\beta} > 0$ only depends on α and β.

Combining now inequalities (68) and (69), we find

$$\|Cz\|_Y^2 \geqslant \delta_\Gamma \sum_{p^2+q^2=m^2+n^2} \frac{q^2}{p^2+q^2} |z_{p,q}|^2 \geqslant \frac{\delta_\Gamma}{\lambda_{m,n}} \|z\|_X^2,$$

which achieves the proof. Here $\delta_\Gamma := \frac{4}{\pi^2}\tilde{\delta}_{\alpha,\beta}$ only depends on Γ.

□

Proof of Theorem 5.3. The result of the theorem is a direct consequence of Lemma 4.1, Theorem 4.1, and Lemma 5.1. We finally obtain that C is spectrally coercive with $\widetilde{\varepsilon}(\lambda) = \frac{1}{2}\left(\frac{2M}{\psi(\lambda)} + 1\right)^{-1}$ and $\widetilde{\psi}(\lambda) = \frac{1}{4}\psi(\lambda)$, which finishes the proof.

□

Remark 5.1. *We observe that the result of Theorem 5.2 based on clever analysis of Fourier series derived in* [8]*, is indeed a particular case of Theorem 5.3 ($\alpha = 0$ and $\beta = \pi$) obtained from Ingham type inequalities.*

Appendix

Let $\chi \in C_0(\mathbb{R})$ be a cut off function with a compact support in $(-1,1)$ given by

$$\chi(s) = (1-|s|)e^{-2|s|}\mathbb{1}_{(-1,1)}.$$

Then we have the following result.

Proposition 5.3. *The function $\chi(s)$ satisfies*

$$\chi \in H_0^1(-1,1),\ \frac{\kappa_1}{1+\tau^2} \leqslant |\widehat{\chi}(\tau)| \leqslant \frac{\kappa_2}{1+\tau^2}, \tau \in \mathbb{R},$$

where $\kappa_1 > \kappa_2$ are two fixed constants.

Proof. Since $|\widehat{\chi}(\tau)|$ is even we shall prove the inequality only for $\tau \in \mathbb{R}_+$.

A forward computation gives

$$\widehat{\chi}(\tau) = \frac{2}{1+\tau^2} + 2\Re\left(\frac{1-e^{-(1+i\tau)}}{(1+i\tau)^2}\right).$$

Then

$$|\widehat{\chi}(\tau)| \leqslant \frac{6}{1+\tau^2}.$$

On the other hand, we have

$$\widehat{\chi}(\tau) = \int_{\mathbb{R}} \frac{2}{1+(s-\tau)^2} \mathrm{sinc}^2\left(\frac{s}{2}\right) ds.$$

Using the estimate $\mathrm{sinc(s)} \geqslant \frac{2}{\pi}$ for $s \in (0, \frac{\pi}{2})$, we get

$$
\begin{aligned}
\widehat{\chi}(\tau) &\geqslant \frac{4}{\pi}\int_{-\frac{1}{4}}^{\frac{1}{4}} \frac{1}{1+(s-\tau)^2}ds \\
&\geqslant \frac{4}{\pi}(\arctan(\frac{1}{4}-\tau)+\arctan(\frac{1}{4}+\tau)) = \frac{4}{\pi}\arctan\left(\frac{\frac{1}{2}}{\frac{15}{16}+\tau^2}\right) \\
&\geqslant \frac{4}{\pi}\left[\frac{\frac{1}{2}}{\frac{15}{16}+\tau^2} - \frac{1}{3}\left(\frac{\frac{1}{2}}{\frac{15}{16}+\tau^2}\right)^3\right] \\
&\geqslant \frac{\frac{4}{3\pi}}{\frac{15}{16}+\tau^2} \\
&\geqslant \frac{\frac{4}{3\pi}}{1+\tau^2},
\end{aligned}
$$

which finishes the proof. □

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